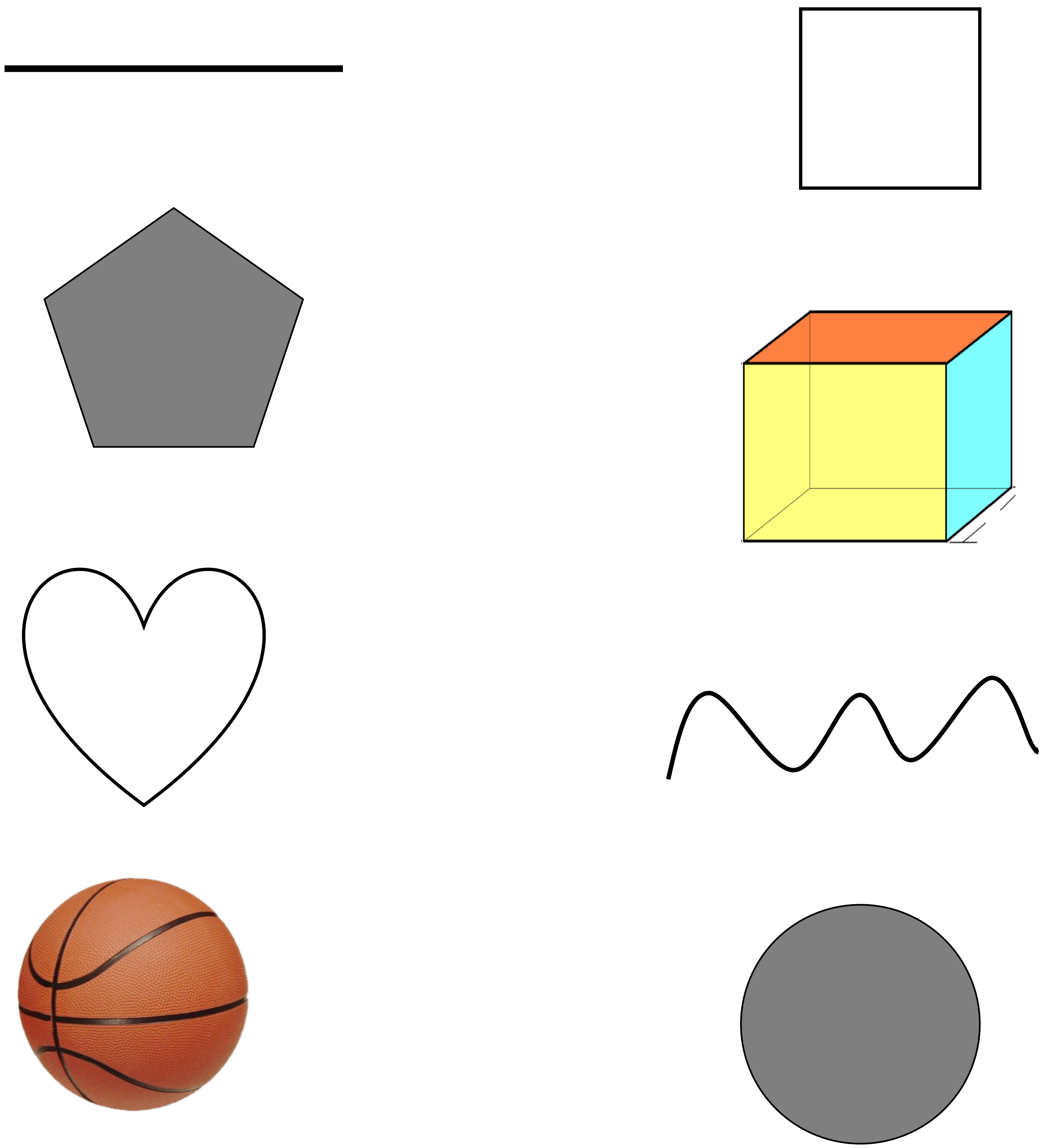
Euler Characteristic

In topology, a branch of mathematics, we study shapes that are not rigid. We imagine that all shapes are made of rubber, so two shapes are the same type if one can be stretched or molded into the other, without breaking it.

Here are some example of shapes. Which of them do you think are the same type, topologically? Connect the same types of shapes with lines.



In 1752, the mathematician Leonard Euler discovered a simple formula that could tell shapes apart. Today we call this formula the **Euler characteristic**. This is what, in mathematics, we call an **invariant**.

If two shapes have **different** Euler Characteristics, then they are **different**.
If two shapes are the **same**, then they have the **same** Euler characteristic.
Beware: If two shapes have the **same** Euler characteristic, they do **may or may not** be the same.

Euler Characteristic in 2-dimensions

All you need to compute the Euler characteristic is the number of vertices (V) and the number of edges (E). Then the Euler characteristic is given by V-E. It's that simple!

Example 1.1

V = number of vertices (dots) = 2

E = number of edges = 1

V – E = 2 -1 = 1. Thus, the Euler characteristic is 1

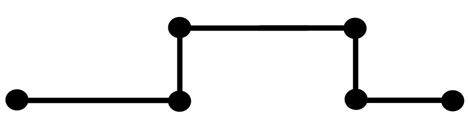
Example 1.2

V = ________

E = ________

V – E = ________

Example 1.2



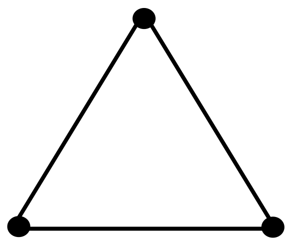
V = ________

E = ________

V – E = ________

Notice that all of these lines have the same Euler characteristic! This makes sense, since they are all the same type of shape.

Example 1.4

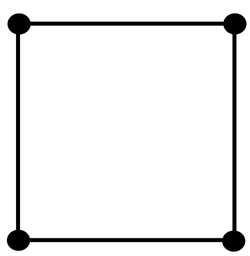


V = ________

E = ________

V – E = ________

Example 1.5

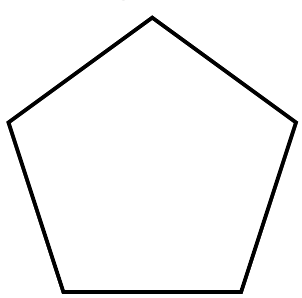


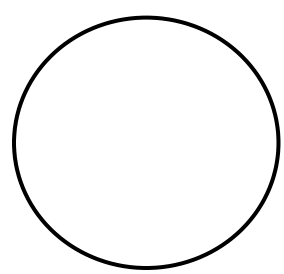
V = ________

E = ________

V – E = ________

Without calculating V or E, what is the Euler characteristic of the following shapes?



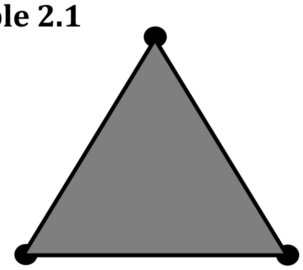


Since all the shapes on this page have Euler characteristic 0, they are **different** than the shapes on the first page.

Euler Characteristic in 3-dimensions

To use this formula in 3 dimensions, you need to know one extra piece of information: the number of faces of the shape. Count the number of vertices (V), the number of edges (E), and the number of faces (F). Then the Euler characteristic is given by V – E + F. Let's try some examples.

Example 2.1



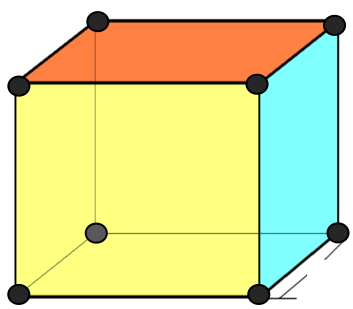
V = ________

E = ________

F = ________

V – E + F = ________

Example 2.2



V = ________

E = ________

F = ________

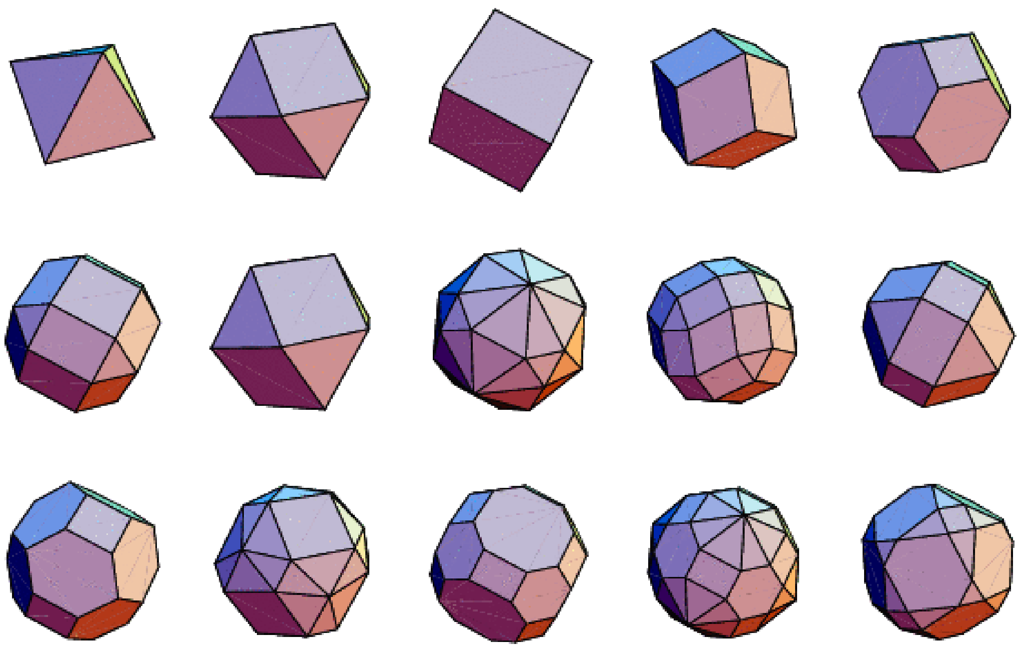
V – E + F = ________

Based on their Euler characteristics, are these shapes the same?

Notice, the Euler characteristic of example 2.1 is the same as the Euler characteristic of the lines in examples 1.1, 1.2, 1.3. This does **not** mean that the triangle and those lines are the same. Obviously they are different types of shapes.

It is usually easy to visualize and draw 2-dimensional and 3-dimensional shapes and so its usually easy to tell which shapes are the same and which shapes are different, without computing the Euler characteristic. Many mathematicians, however, are interested in shapes that are **4-dimensional, 5-dimensional, 6-dimensional, etc.** These are impossible to visualize, so formulas like the Euler characteristic (called **invariants)** are extremely useful in telling shapes apart.

Euler's original finding in 1752 was that all simple 3 dimensional shapes, called **polyhedra**, have the same Euler characteristic. A polyhedron is a 3 dimensional shape that has vertices, edges, and faces such each edge connects two faces.



Notice that, if these shapes were made of rubber, we could reshape them to look like each other. Thus, they are topologically the same type of shape. In particular, they are the same as example 2.2.

Since they are all the same, they should all have the same Euler characteristic. What is it?